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SOLUTION BY W. E. MILNE, Bowdoin College.

Applying successively the formula $\sin 2\theta = 2 \cos \theta \sin \theta$ we obtain

$$\sin x = 2 \cos \frac{x}{2} \sin \frac{x}{2}, \quad = 4 \cos \frac{x}{2} \cos \frac{x}{4} \sin \frac{x}{4}, \quad = 8 \cos \frac{x}{2} \cos \frac{x}{4} \cos \frac{x}{8} \sin \frac{x}{8}, \quad \dots,$$

and in general

$$(1) \quad \frac{\sin x}{2^n \sin (x/2^n)} = \cos \frac{x}{2} \cos \frac{x}{2^2} \cos \frac{x}{2^3} \cdots \cos \frac{x}{2^n}.$$

Now $\lim_{n \rightarrow \infty} 2^n \sin(x/2^n) = x$. Since equation (1) holds for every positive integral value of n ,

and the left-hand side approaches a limit $\frac{\sin x}{x}$ as n becomes infinite, it follows that the right-hand side also approaches a limit, and the limits are equal. Therefore,

$$\frac{\sin x}{x} = \cos \frac{x}{2} \cos \frac{x}{2^2} \cos \frac{x}{2^3} \cos \frac{x}{2^4} \cdots.$$

Also solved by E. B. WILSON, PAUL CAPRON, HORACE OLSON, C. A. NICKLE, A. H. HOLMES, H. S. BROWN, E. H. VANCE, NORMAN ANNING, J. A. BULLARD, J. J. GINSBURG, C. K. ROBBINS, H. L. AGARD, H. R. HOWARD, O. S. ADAMS, I. A. BENNETT, J. A. CAPARO, and the PROPOSER.

GEOMETRY.

479. Proposed by NATHAN ALTSHILLER, University of Colorado.

Find the locus of the point whose polars (polar planes) with respect to two given conics (quadrics), are perpendicular to each other.

SOLUTION BY THE PROPOSER.

A. POLARS WITH RESPECT TO CONICS.

If P is a point of the required locus, its polars p_1, p_2 with respect to the two given conics $(\Sigma_1), (\Sigma_2)$ meet the line at infinity in two points Q_1, Q_2 separated harmonically by the cyclical points $\mathfrak{J}, \mathfrak{J}'$ at infinity. The pole, with respect to (Σ_1) , of any line passing through Q_1 lies on the polar q_1 of Q_1 with respect to (Σ_1) ; and the pole, with respect to (Σ_2) , of any line passing through Q_2 lies on the polar q_2 of Q_2 with respect to (Σ_2) . When Q_1 varies on the line at infinity, q_1 turns about the center C_1 of (Σ_1) , the range (Q_1) and the pencil (q_1) being projective. Similarly for the range (Q_2) and the pencil (q_2) . The points Q_1, Q_2 being a couple of conjugate elements in the involution whose double elements are the cyclical points $\mathfrak{J}, \mathfrak{J}'$, we have

$$(q_1 \cdots) \wedge (Q_1 \cdots) \wedge (Q_2 \cdots) \wedge (q_2 \cdots)$$

Hence: *The locus of the point, whose polars, with respect to two given conics, are perpendicular to each other, is, in general, a conic passing through the centers of the given curves.*

I. Two central conics. If (Σ_1) and (Σ_2) are both central conics, let (C) be the circle having for diameter the line C_1C_2 of their centers. The couples of conjugate diameters of (Σ_1) determine an involution (I_1) of points on (C) . Let O_1 denote the pole of this involution. The conic (Σ_2) gives rise to a similar involution (I_2) with O_2 as its pole. Let M be any point of (C) . If M_1, M_2 are the two other points of intersection of (C) with the lines O_1M, O_2M , respectively, the point of intersection P of the diameters C_1M_1, C_2M_2 will belong to the required locus (Σ) , because the polars of P with respect to (Σ_1) and (Σ_2) are parallel to the diameters C_1M and C_2M , respectively conjugate to C_1P, C_2P in the two given conics, and the two former are perpendicular to each other. This affords an expeditious way for the construction of (Σ) .

The points of intersection N, N' of the circle (C) with the line O_1O_2 belong to the required locus (Σ) , as is readily seen by making the point M coincide with each one of them in turn. These two points will necessarily be real, if at least one of the points O_1, O_2 lies inside of (C) .

The double elements of (I_1) on (C) are the points of contact of the tangents to (C) drawn from O_1 . The lines joining C_1 to these points of contact are the double elements of the involution of

conjugate diameters of (Σ_1) , *i. e.*, the asymptotes of (Σ_1) . Hence O_1 will lie outside of (C) if (Σ_1) is an hyperbola, and inside of (C) if it is an ellipse or a circle. Similarly for O_2 .

The conic (Σ) will have a real point at infinity if the lines C_1M_1 , C_2M_2 are parallel. The line C_1C_2 being a diameter of (C) , the quadrilateral $C_1M_1C_2M_2$ inscribed in (C) is a rectangle, and M_1M_2 is a diameter, hence $\angle M_1MM_2 = \angle O_1MO_2 = 90^\circ$. The point M is therefore common to (C) and to the circle having O_1O_2 for diameter. The points of intersection of these two circles are necessarily real and distinct when one of the points O_1 , O_2 is inside and the other outside of (C) . They may be real, imaginary, or coincident, when O_1 , O_2 are both inside, or both outside of (C) . Consequently: *The conic (Σ) is, in general, an hyperbola, if one of the given conics is an hyperbola and the other an ellipse. Otherwise (Σ) may be an ellipse, an hyperbola, or a parabola.*

Since the conic (Σ) passes through the four points C_1 , C_2 , N , N' of (C) , the conic can have no other point in common with (C) and remain distinct from this circle. Therefore, if (Σ) is to be a circle distinct from (C) this is only possible, if the points of intersection of O_1O_2 with (C) are the cyclical points at infinity, *i. e.*, O_1O_2 must itself be the line at infinity, and hence the points O_1 , O_2 are points at infinity. But the tangents to (C) drawn from O_1 at infinity touch (C) at two diametrically opposite points, the asymptotes of (Σ_1) are therefore real and orthogonal. Similarly for (Σ_2) . Hence: *The locus of the point whose polars, with respect to two given equilateral hyperbolas, are perpendicular to each other, is, in general, a circle distinct from the circle having for diameter the line of centers of the given conics.*

The reader may prove the following statements: If an asymptote of (Σ_1) is perpendicular to an asymptote of (Σ_2) , the conic (Σ) is tangent to (C) at the point of intersection X of these lines. If X coincides, say, with C_1 , (C) is the osculating circle of (Σ) at this point. If C_1C_2 is an axis for each of the given conics, (Σ) has a double contact with (C) at C_1 , C_2 .

If O_1C_2 , O_2C_1 meet on (C) , to the line C_1C_2 of the pencil (q_1) at C_1 will correspond the line C_2C_1 of the pencil (q_2) at C_2 , so that the line C_1C_2 will be a part of the locus (Σ) , the other part being the line O_1O_2 . Hence: *If the diameters conjugate in the two given conics, respectively, to the line of their centers, are perpendicular to each other, the locus degenerates into two straight lines, one of them being the line of their centers.*

The conic (Σ) will be identical with (C) , if for any point M of (C) the points M_1 , M_2 coincide; but then the lines O_1MM_1 and O_2MM_2 will coincide, which makes it in turn necessary for O_1 , O_2 to coincide, say in O . It is evident that this last condition is also sufficient for the coincidence of (Σ) with (C) . If the involutions (I_1) and (I_2) have the same pole O , the diameters C_1M , C_1M_1 of (Σ_1) which are perpendicular to any couple of conjugate diameters C_2M , C_2M_2 of (Σ_2) , are themselves conjugate in (Σ_1) . Consequently: *If the couples of conjugate diameters of one of the given conics are the perpendiculars dropped from its center upon the couples of conjugate diameters of the other conic, the required locus is the circle (C) .* This will take place, for instance, when the two given conics are both circles, or hyperbolas with mutually perpendicular asymptotes.

If the given conics are concentric, the two projective pencils (q_1) , (q_2) are superposed, and their double elements constitute the required locus. These lines may conveniently be constructed by drawing any circle (C) through the common center C of the given conics and determining the points O_1 and O_2 as above; the lines projecting from C the points of intersection of (C) with the circle having O_1O_2 for its diameter, are the required lines. They are always real when one of the given conics is an ellipse (a circle) and the other an hyperbola. When the points O_1 , O_2 coincide, *i. e.*, when the given conics have the same involution of conjugate diameters, the required locus has no real points.

N. B. In the above discussion of the central conics (Σ_1) , (Σ_2) , only their involutions of conjugate diameters were considered. The locus (Σ) would therefore not change, if (Σ_1) would be replaced by any other conic of the pencil having the same involution of conjugate diameters as (Σ_1) . Similarly for (Σ_2) . For instance, an hyperbola may be replaced by any hyperbola having the same asymptotes: a circle, by any other concentric circle.

II. A parabola and a central conic. The locus (Σ) will pass through the center C_1 of (Σ_1) and through the point at infinity of the parabola (Σ_2) . If q_1 , q_1' are any two conjugate diameters of (Σ_1) , t_2 the tangent to (Σ_2) perpendicular to q_1' , the diameter q_2 of (Σ_2) conjugate to t_2 will meet q_1 in a point of (Σ) . If d_1 , d_1' are the two conjugate diameters of (Σ_1) , of which one, say d_1' , is perpendicular to the axis a_2 of (Σ_2) , the point at infinity of d_1 belongs to (Σ) , as may be readily seen. Hence: *The required locus is, in general, an hyperbola passing through C_1 , whose asymptotic directions are a_2 and d_1 .*

The two asymptotic directions of (Σ) will be perpendicular to each other, when and only when the conjugate diameter d_1 of d_1' coincides with d_1' , *i. e.*, when d_1' is an asymptote of (Σ_1) . Hence:

The locus will be an equilateral hyperbola, if the axis of the parabola is perpendicular to an asymptote of the other conic.

The two points at infinity of the locus (Σ) will coincide, if d_1 is parallel to a_2 , i. e., if d_1' is an axis of (Σ_1). Hence: *The required locus will be a parabola, if the axis of the given parabola is parallel to one of the axes of the other given conic.*

If the central conic is a circle, the locus is always a parabola.

In the construction above, if for the diameter g_2 is taken the one n_2 passing through C_1 , the corresponding diameter n_1 of (Σ_1) is the tangent to (Σ) at this point. The diameter n_1 will coincide with n_2 , if the conjugate directions of n_2 in (Σ_1) and (Σ_2), respectively, are perpendicular to each other. The line n_2 is then a part of (Σ). Hence: *If the directions conjugate in the two given conics to the line of their centers, are perpendicular to each other, the locus degenerates into two straight lines, one of which is their line of centers.*

The reader may discuss the case when the axis of the parabola passes through the center of the other given conic.

III. Two parabolas. The required locus passes through the points at infinity of the given parabolas (Σ_1), (Σ_2), the conic (Σ) is therefore always an hyperbola. Draw the tangent to (Σ_2) which is perpendicular to the axis of (Σ_1). The diameter of (Σ_1) which passes through the point of contact, is one of the asymptotes of (Σ). The other asymptote is found in the same way starting with (Σ_1). The hyperbola (Σ) will be equilateral, if the axes of the parabolas are perpendicular. If these axes are parallel, their common point at infinity is the center of the two superposed projective pencils (q_1), (q_2), whose double elements constitute the required locus.

B. POLAR PLANES WITH RESPECT TO QUADRICS.

If P is a point of the required locus, its polar planes π_1 , π_2 with respect to the two given quadrics (Σ_1), (Σ_2), meet the plane at infinity in two lines p_1 , p_2 , which are conjugate with respect to the imaginary circle at infinity (γ), i. e., p_2 passes through the pole M of p_1 with respect to (γ). The poles, with respect to (Σ_1) of all the planes passing through p_1 , are on the line m_1 which is the conjugate to p_1 with respect to (Σ_1); the poles, with respect to (Σ_2), of all the planes passing through M , are in the polar plane μ_2 of M with respect to (Σ_2). The point P is therefore the point of intersection of the line m_1 with the plane μ_2 . If p varies in the plane at infinity, m_1 describes the bundle of lines (m_1) having for its center the center of (Σ_1), and the two forms (p_1), (m_1) are reciprocal; if the point M varies in the plane at infinity, μ_2 describes a bundle of planes (μ_{21}), having for its center the center of (Σ_2), and the two forms (M) and (μ_2) are reciprocal. The two plane forms (M), (p_1) being reciprocal with respect to (γ), the bundle of lines (m_1) is reciprocal to the bundle of planes (μ_2). Hence: *The locus of the point whose polar planes, with respect to two given quadrics, are perpendicular to each other, is, in general, a quadric passing through the centers of the given surfaces.*

Also solved analytically by FLORENCE P. LEWIS.

480. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

Of equal quadrilaterals on the same base, that which has the least perimeter must have the angles not adjacent to the base equal to each other.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

We shall first prove a different theorem and then show it is equivalent to the above. "Of isoperimetric quadrilaterals on the same base, that which has the greatest area must have the angles not adjacent to the base equal to each other." Call the vertices $ABCD$, the given base being AD . The maximum quadrilateral must have $AB = BC = CD$, since, if the triangle ABC is a maximum, it is isosceles. Again, it must be inscriptible in a circle and hence is an isosceles trapezoid with angle B equal to angle C .

Now suppose we could find another quadrilateral, $AB'C'D$, having the same area as $ABCD$, but a smaller perimeter. It is clear that we can then construct a third quadrilateral, e. g., by moving the vertex B' to B'' , so that $AB''C'D$ has a larger area than $AB'C'D$ or $ABCD$, but the same perimeter as $ABCD$. But this is impossible. Also, there can be no other quadrilateral having the same area and perimeter. Hence, $ABCD$ has the least perimeter, for the given area, as well as the most area for a given perimeter.